The Calculus Underlying Aggregate-Only Stop Loss Programs

November 9, 2015

Background

The pricing for low attachment point aggregate-only stop loss programs necessarily involves integrating probability density functions. This probability density function is determined through an actuarial and underwriting process, taking into account the size of the group, the duration year in question, the demographics of the group, and the collective health status of the members of the group.

The actuarial and underwriting process produces an expected claim amount for the group's rating year. This expected claims figure can be expressed as a single figure, a set of composite rates, or even as single age and family status rates.

The mathematical result of the following calculus exercise provides pricing factors which, when applied to the expected claims rates, produces the framework for the aggregate-only rate structure.

Preliminary Constructs

A random variable that is a given group's outcome for a given rating year's actual-to-expected plan claims ratio:

$$X, for\ all\ x \geq 0$$

The probability density function for X:

$$f_X(x), for\ all\ x \geq 0$$

The cumulative probability distribution function for X:

$$F_X(x) = \int_0^x f_X(t)dt$$

As a probability density function for X, its integral is naturally 1:

$$\int_0^\infty f_X(x)dx = 1$$

Additionally, since X is an actual to expected ratio, its expected value is 1:

$$\int_0^\infty x f_X(x)dx = 1$$

The Essential Integration of the Density Function

The aggregate stop loss policy has an attachment point, which is fully funded by the employer. Its factor as a ratio to plan expected claims is denoted:

$$a, where\ a > 0$$

The probability of an aggregate claim is:

$$1 - F_X(a) = 1 - \int_0^a f_X(x)dx = \int_a^\infty f_X(x)dx$$

Three derivative random variables, each a mixed function of X:

$$\begin{array}{lll} Aggregate\ Stop\ Loss\ Claim & SLC(X) & = \begin{cases} 0 & x \le a \\ X - a & x > a \end{cases} \\ Employer\ Funded\ Claim & EFC(X) & = \begin{cases} X & x \le a \\ a & x > a \end{cases} \\ Employer\ Funding\ Surplus & EFS(X) & = \begin{cases} a - X & x \le a \\ 0 & x > a \end{cases} \end{array}$$

Inspection shows that the stop loss claim and the employer funded claim partition the full claim since:

$$SLC(X) + EFC(X) = \left.\begin{cases} 0 + X & x \le a \\ X - a + a & x > a \end{cases}\right\} = X$$

The strucure for the three separate integrations that produce pricing factors:

$$\begin{array}{lll} Net\ Aggregate\ Stop\ Loss\ Premium\ Factor & NP & = \int_0^\infty SLC(X) f_X(x)dx \\ Expected\ Employer\ Funded\ Claim\ Factor & EF & = \int_0^\infty EFC(X) f_X(x)dx \\ Expected\ Employer\ Funding\ Surplus\ Factor & ES & = \int_0^\infty EFS(X) f_X(x)dx \end{array}$$

The probability of an employer funding surplus:

$$F_X(a) = \int_0^a f_X(x)dx$$

The probability of an aggregate claim occurring:

$$1 - F_X(a) = \int_a^\infty f_X(x)dx$$

The net premium factor for the stop loss policy is the expected value of the stop loss claim ratio. Each of these expressions capture the relation:

$$NP = \int_0^\infty SLC(X) f_X(x)dx$$
$$NP = \int_0^a (0) f_X(x)dx + \int_a^\infty (x - a) f_X(x)dx$$
$$NP = \int_a^\infty (x - a) f_X(x)dx$$
$$NP = \int_a^\infty x f_X(x)dx - a \int_a^\infty f_X(x)dx$$
$$NP = \int_a^\infty x f_X(x)dx - a\big(1 - F_X(a)\big)$$

The expected employer funding factor is the expected value of the employer funded claim ratio, expressed in different ways:

$$EF = \int_0^\infty EFC(X) f_X(x)dx$$
$$EF = \int_0^a x f_X(x)dx + \int_a^\infty a f_X(x)dx$$
$$EF = \int_0^a x f_X(x)dx + a\big(1 - F_X(a)\big)$$

The expected employer surplus factor is the expected value of the employer funding surplus ratio:

$$
\begin{aligned}
ES &= \int_0^\infty EFS(X) f_X(x)dx \\
ES &= \int_0^a (a-x) f_X(x)dx + \int_a^\infty (0) f_X(x)dx \\
ES &= \int_0^a (a-x) f_X(x)dx \\
ES &= \int_0^a a f_X(x)dx - \int_0^a x f_X(x)dx \\
ES &= aF_X(a) - \int_0^a x f_X(x)dx
\end{aligned}
$$

The Relationship among the Resulting Rating Factors

The net premium factor and the expected claims factor partition the plan claims; summing to one:

$$NP + EF = 1$$

The attachment point factor is partitioned into the expected claims factor and the expected surplus:

$$EF + ES = a$$

The net premium factor and the attachment factor, in other words, the amount the employer funds before any gain or expense is considered, exceed one by the expected surplus factor:

$$NP + a = 1 + ES$$

The Pricing Objectives

With the specification of two pricing objectives, a premium expense load, and fees for third parties, these factors, with their associated identities, form the key constructs for the aggregate pricing.

The frequency pricing objective, f, is set as the probability of an aggregate claim, normally set to 2/3. The attachment point factor is determined by solving the equation for a:

$$1 - f = F_X(a) = \int_0^a f_X(x)dx = 1 - \int_a^\infty f_X(x)dx$$

$$a = F_X^{-1}(1 - f)$$

The underwriting gain objective, g, is set as a proportion of ground-up claims, normally .08, and the premium expense load, e, is set as a proportion of gross premium, normally in the range of .08 to .12. The gross premium factor is straightforward as:

$$GP = \frac{NP + g}{1 - e}$$

And the gross profit ratio is:

$$\frac{g}{GP} = \frac{(1 - e)}{1 + \frac{NP}{g}}$$

<u>The Final Rates</u>

The fully funded aggregate stop loss arrangement is fully constructed by applying these resultant measures to the plan claim rates developed by the underwriting process:

$$\begin{aligned} &Expected\ Funding\ Surplus\ Rates &&= &&GP\ x\ Plan\ Claim\ Rates \\ &Funding\ Rates &&= &&a\ x\ Plan\ Claim\ Rates \\ &Expected\ Funding\ Surplus\ Rates &&= &&ES\ x\ Plan\ Claim\ Rates \end{aligned}$$

Additionally, third party fees are expressed in a similar rate form.